

PERIODS OF HODGE STRUCTURES AND SPECIAL VALUES OF THE GAMMA FUNCTION

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ABSTRACT. At the end of the 70s, Gross and Deligne conjectured that periods of geometric Hodge structures with multiplication by an abelian number field are products of values of the gamma function at rational numbers, with exponents determined by the Hodge decomposition. We prove an alternating variant of this conjecture for the cohomology groups of smooth, projective varieties over $\overline{\mathbb{Q}}$ acted upon by a finite order automorphism, thus improving previous results of Maillot and RöSSLer. The proof relies on a product formula for periods of integrable connections due to Saito and Terasoma.

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0. INTRODUCTION AND OVERVIEW

This paper continues a series of works by Weil [Wei76], Gross [Gro78], Shimura [Shi79], Anderson [And82], Deligne [Del82], Colmez [Col93] and, more recently, Maillot and

Rössler [MR04], aiming to understand the relations between periods of certain algebraic varieties (or motives) over $\overline{\mathbb{Q}}$ —the algebraic closure of \mathbb{Q} in \mathbb{C} —and values of the gamma function at rational numbers. Recall that the latter is defined by the convergent integral

$$\Gamma(s) = \int_0^{+\infty} t^{s-1} e^{-t} dt$$

when $\operatorname{Re}(s) > 0$, and extended to a nowhere vanishing meromorphic function on \mathbb{C} , with simple poles at non-positive integers, by means of the equation $\Gamma(s+1) = s\Gamma(s)$.

0.1. Euler's formula. The oldest instance of such a relation is Euler's formula

$$\int_0^1 t^{a-1} (1-t)^{b-1} dt = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, \quad \operatorname{Re}(a), \operatorname{Re}(b) > 0,$$

whose left hand side can be interpreted, for rational values of a and b , as a period of a Fermat curve. Indeed, leaving aside some trivial cases, we can assume that $a = r/d$ and $b = s/d$ for some integers $d \geq 3$ and $0 < r, s < d$ such that $r + s \neq d$. Consider the cyclotomic field $k = \mathbb{Q}(\zeta)$, for a fixed primitive d -th root of unity ζ , and let $C \subset \mathbb{P}^2$ be the curve $x^d + y^d = z^d$, regarded as a variety over k . The group $G = \mu_d^3 / (\text{diag})$ acts on C by coordinate-wise multiplication, and this action decomposes the algebraic de Rham cohomology $H_{dR}^1(C/k)$ into one-dimensional representations. As it turns out, the class of $\omega_{r,s} = x^{r-1} y^{s-d} dx$ spans the subspace where G acts through the character $\chi_{r,s}(\zeta^i, \zeta^j) = \zeta^{ir+j^s}$, and, for a suitable chosen topological cycle $\gamma \in H_1(C_{\mathbb{C}}^{\text{an}}, \mathbb{Q})$, one has:

$$(1) \quad \int_{\gamma} \omega_{r,s} \sim_{k^\times} \int_0^1 t^{a-1} (1-t)^{b-1} dt,$$

where \sim_{k^\times} indicates that both terms agree up to a non-zero factor in k [Lan82, Ch. 5].

From the point of view of this article, Euler's formula may be understood as an expression for the periods of rank one local systems over the punctured Riemann sphere $\mathbb{P}^1(\mathbb{C}) - \{0, 1, \infty\}$ purely in terms of the local monodromies around the missing points. To make this more precise, observe that the morphism $\pi : \mathbb{P}^2 \rightarrow \mathbb{P}^2$ obtained by raising all coordinates to the d -th power realizes C as a G -cover of \mathbb{P}^1 , unramified outside $D = \{0, 1, \infty\}$. By transport of structure, G acts on the direct image of the exterior differential $\mathcal{O}_C \rightarrow \Omega_C^1$, splitting it into a direct sum of rank one logarithmic connections $\nabla_\chi : \mathcal{E}_\chi \rightarrow \mathcal{E}_\chi \otimes \Omega_{\mathbb{P}^1}^1(\log D)$ indexed by the characters χ of G . The analytic horizontal sections of \mathcal{E}_χ restricted to $U = \mathbb{P}^1 - D$ define a complex local system $\ker(\nabla_\chi^{\text{an}})$ on $U_{\mathbb{C}}^{\text{an}}$, which canonically identifies with the eigensheaf of $(\pi_* \mathbb{C})|_{U^{\text{an}}}$ corresponding to χ . Thus one gets a local system V_χ of k -vector spaces on $U_{\mathbb{C}}^{\text{an}}$ such that $\ker(\nabla_\chi^{\text{an}}) \simeq V_\chi \otimes_k \mathbb{C}$. At the level of cohomology, integration along chains induces “period isomorphisms”

$$\mathbb{H}^1(U, \mathcal{E}_\chi \xrightarrow{\nabla_\chi} \Omega_U^1) \otimes_k \mathbb{C} \simeq H^1(U_{\mathbb{C}}^{\text{an}}, V_\chi) \otimes_k \mathbb{C},$$

whose coefficients with respect to convenient k -basis are nothing other than the integrals in (1). Now an easy computation shows that the residues of the connection $\nabla_{\chi_{r,s}}$ at the singular points 0, 1 and ∞ are a , b and $-a - b$ respectively, the rational numbers at which the gamma function is evaluated on the right hand side of Euler's formula.

Following an idea of Bloch [Blo05], we will use a far-reaching generalization of Euler's formula, the Saito-Terasoma theorem, to compute periods of varieties with finite order automorphisms.

0.2. CM abelian varieties. Another main source comes from abelian varieties with complex multiplication. By this we mean triples (A, F, ι) consisting of an abelian variety A of dimension n over $\overline{\mathbb{Q}}$, a CM number field F of degree $2n$ and a ring morphism $\iota: F \hookrightarrow \text{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$. The induced action on the $\overline{\mathbb{Q}}$ -vector space of holomorphic differentials $H^0(A, \Omega_A^1)$ defines a CM type of F , that is, a subset $\Phi \subset \text{Hom}(F, \mathbb{C})$ composed of n pairwise non-conjugate complex embeddings. To these data, one associates periods as follows: for each $\sigma \in \text{Hom}(F, \mathbb{C})$, choose a non-zero algebraic de Rham cohomology class $\omega_\sigma \in H_{\text{dR}}^1(A/\overline{\mathbb{Q}})$ such that $\iota(f)^* \omega_\sigma = \sigma(f) \omega_\sigma$ for all $f \in F$. Besides, let $\gamma \in H_1(A_{\mathbb{C}}^{\text{an}}, \mathbb{Q})$ be any topological one-cycle not homologous to zero. Then the complex number

$$(2) \quad P_\sigma(A, F, \iota) := \int_\gamma \omega_\sigma$$

is non-zero, and its class modulo $\overline{\mathbb{Q}}^\times$ does not depend on the choices of ω_σ and γ . What is more, it depends only on F , σ and Φ , since all abelian varieties with the same CM type are isogenous over $\overline{\mathbb{Q}}$ [Mum70, §22]; it is, therefore, more accurate to write $P_\sigma(F, \Phi)$. Shimura found monomial relations between periods corresponding to different choices of the defining data, the easiest one being $P_\sigma(F, \Phi) \cdot P_{\bar{\sigma}}(F, \Phi) \sim_{\overline{\mathbb{Q}}^\times} 2\pi i$, from which (2) can be computed using holomorphic differentials alone [Shi79, §1].

Whenever F is an abelian extension of \mathbb{Q} , Anderson proved that the period $P_\sigma(F, \Phi)$ agrees, up to an algebraic factor, with the exponential of a certain \mathbb{Q} -linear combination of logarithmic derivatives at $s = 0$ of Dirichlet L-functions [And82]. Later on, Colmez used p -adic periods and the Faltings height to get an actual identity of real numbers, which he conjectured to hold true also in the non-abelian case, provided one replaces Dirichlet by Artin L-functions [Col93]. Only the case of certain quartic non-Galois CM fields has been settled so far [Ya10a, Ya10b, Ya13]. Back to the abelian realm, combining Anderson's theorem with a classical formula of Hurwitz, one can express $P_\sigma(F, \Phi)$ as a product of values of the gamma function at rational numbers, with denominator the conductor of F , and exponents depending only on the CM type. For example, if $\sigma: F \hookrightarrow \mathbb{C}$ is an imaginary quadratic field of discriminant $-d$, one gets back the celebrated Lerch-Chowla-Selberg formula

$$(3) \quad P_\sigma(F, \sigma) \sim_{\overline{\mathbb{Q}}^\times} \sqrt{\pi} \prod_{a=1}^{d-1} \Gamma\left(\frac{a}{d}\right)^{\frac{w\chi(a)}{4h}},$$

where w is the number of roots of unity in F , h its class number and $\chi: (\mathbb{Z}/d)^\times \rightarrow \{\pm 1\}$ the Dirichlet character attached to F .

The identity (3) has a rich history, for which we refer the reader to [Sch88, Ch 3, §2] and the references therein. Let us just mention that it was first proved by analytic methods, namely the Kronecker limit formula. A major obstacle when trying to give a geometric proof is that complex multiplication on elliptic curves is not given by automorphisms, except when $d = 3$ or 4 ¹. Gross circumvented the problem by deforming a power of the elliptic curve to a simple quotient of the Jacobian of the Fermat curve $x^d + y^d = z^d$, for which the techniques of the previous paragraph are at disposal [Gro78]. This idea has proved extremely fruitful: besides its central role in the works by Anderson and Colmez already quoted, it inspired Deligne's proof that all Hodge cycles on abelian varieties are absolute Hodge [Del82]. As a striking application, one can show that the transcendence degree of the field generated by the periods of a CM abelian variety of

¹Another (more serious) manifestation of the same phenomenon is the lack of any proof of algebraicity of the Weil classes outside these two discriminants...

dimension n is less or equal than $n + 1$ [Del80, §4]. For $n = 1$, Chudnovsky had already proved that equality holds, a result from which the transcendence of the values $\Gamma(1/3)$ and $\Gamma(1/4)$, as well as their respective algebraic independence with π , follows when combined with the Lerch-Chowla-Selberg formula [Chu78].

In a different line of thought, Köhler and Rössler gave a new proof of Colmez's theorem based on their Lefschetz fixed point formula in Arakelov geometry [KR01]. As a starting point, they induce the complex multiplication from A to a suitable power A^r , where it is now given by automorphisms. But, instead of shifting out the computation onto another variety, they directly consider an equivariant line bundle on A^r , whose arithmetic degree computes the Faltings height [KR03]. Alternatively, one can recover the periods from the arithmetic degree of the de Rham complex. As noticed by Maillot and Rössler [MR04, Thm. 2], this greatly simplifies the proof of [KR03], since the most involved term of the Lefschetz fixed point formula –the equivariant analytic torsion– vanishes in this case for symmetry reasons. More importantly, the same strategy works for any arithmetic variety endowed with a finite order automorphism, yielding new relations between periods and logarithmic derivatives of Dirichlet L-functions (see §0.4 below). The Bourbaki seminar [Sou07] contains a nice survey of these ideas.

0.3. The Gross-Deligne conjecture. Motivated by his approach to the Lerch-Chowla-Selberg formula, Gross conjectured² in [Gro78] that the results for CM abelian varieties extend to any geometric Hodge structure with “abelian multiplication”. Let us recall the precise statement and see how the previous examples fit into the picture.

Let (H, F, ι) be a triple consisting of a pure rational Hodge structure H of weight m , a number field F of degree $\dim_{\mathbb{Q}} H$ and a ring morphism $\iota : F \hookrightarrow \text{End}_{HS}(H)$ from F to the endomorphisms of H as Hodge structure. Then there exists a canonical decomposition

$$(4) \quad H \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{\sigma \in \text{Hom}(F, \mathbb{C})} H_{\sigma},$$

where H_{σ} stands for the one-dimensional complex vector space $H \otimes_{F, \sigma} \mathbb{C}$. As F acts by morphisms of Hodge structures, each H_{σ} is contained in a unique piece $H^{p, q}$ of the Hodge decomposition of $H \otimes_{\mathbb{Q}} \mathbb{C}$. Let $p : \text{Hom}(F, \mathbb{C}) \rightarrow \mathbb{Z}$ be the function assigning to σ the p -Hodge type of H_{σ} . By Hodge symmetry, $p(\sigma) + p(\bar{\sigma}) = m$.

To define periods, we shall further assume that H is a sub-Hodge structure of the Betti cohomology $H_B^m(X) := H^m(X_{\mathbb{C}}^{\text{an}}, \mathbb{Q})$ of a smooth, projective variety X over $\overline{\mathbb{Q}}$ ³, and that the H_{σ} are defined over $\overline{\mathbb{Q}}$ for the “de Rham structure” stemming from the isomorphism

$$(5) \quad \rho^m : H_{\text{dR}}^m(X/\overline{\mathbb{Q}}) \otimes_{\overline{\mathbb{Q}}} \mathbb{C} \xrightarrow{\sim} H_B^m(X) \otimes_{\mathbb{Q}} \mathbb{C}.$$

This amounts to say that, for each $\sigma \in \text{Hom}(F, \mathbb{C})$, there exists a class $\omega_{\sigma} \in H_{\text{dR}}^m(X/\overline{\mathbb{Q}})$ such that $\rho^m(\omega_{\sigma} \otimes_{\overline{\mathbb{Q}}} 1_{\mathbb{C}})$ is a basis of H_{σ} . Now, if we let $\gamma \in H^{\vee}$ be any non-zero element of the dual of the \mathbb{Q} -vector space H , the number $(\gamma \otimes_{\mathbb{Q}} 1_{\mathbb{C}})(\omega_{\sigma})$ is independent of the choices of ω_{σ} and γ , up to multiplication by a non-zero algebraic number. Viewing γ as a cycle in $H_m(X_{\mathbb{C}}^{\text{an}}, \mathbb{Q})$, it is nothing other than $\int_{\gamma} \omega_{\sigma}$.

Definition 1 ($P_{\sigma}(H)$). The period $P_{\sigma}(H) \in \mathbb{C}^{\times}/\overline{\mathbb{Q}}^{\times}$ is the class modulo $\overline{\mathbb{Q}}^{\times}$ of $(\gamma \otimes_{\mathbb{Q}} 1_{\mathbb{C}})(\omega_{\sigma})$ for any choice of ω_{σ} and γ as above.

²As he says, “the precise version of this conjecture was suggested to me by Deligne”, hence the name.

³For instance, this follows automatically whenever F is a CM number field and the Hodge structure H is polarizable and effective [Abd05, Thm. 3].

Suppose finally that F is an abelian extension of \mathbb{Q} of conductor d , so F embeds into the cyclotomic field $\mathbb{Q}(\mu_d)$. Identifying $\text{Gal}(F/\mathbb{Q})$ with $\text{Hom}(F, \mathbb{C})$ through the fixed complex embedding of F , one gets a surjective map $\varphi : (\mathbb{Z}/d)^\times \rightarrow \text{Hom}(F, \mathbb{C})$. Given an element $\lambda \in (\mathbb{Z}/d)^\times$, we will write $P_\lambda(H)$ for the corresponding period $P_{\varphi(\lambda)}(H)$. Besides, the composition of φ and p gives a function, still denoted $p : (\mathbb{Z}/d)^\times \rightarrow \mathbb{Z}$, which now satisfies $p(\lambda) + p(-\lambda) = m$ since $-1 \in (\mathbb{Z}/d)^\times$ acts by complex conjugation. By [Del77, Lemma 6.12], there exist a function $\varepsilon : \mathbb{Z}/d \rightarrow \mathbb{Q}$ such that, for all $\lambda \in (\mathbb{Z}/d)^\times$,

$$(6) \quad p(\lambda) = \frac{1}{d} \sum_{a=1}^{d-1} \varepsilon(a) \langle a\lambda \rangle,$$

where $\langle x \rangle$ stands for the representative of x between 0 and $d-1$.

Before stating the Gross-Deligne conjecture [Gro78, p. 205], let us summarize all the assumptions of the previous paragraphs:

Hypothesis H. *The triple (H, F, ι) consists of*

- (a) *a sub-Hodge structure $H \subset H_B^m(X)$ of the Betti cohomology of some smooth, projective variety X over $\overline{\mathbb{Q}}$,*
- (b) *an abelian number field F of degree $\dim_{\mathbb{Q}} H$,*
- (c) *a ring morphism $\iota : F \hookrightarrow \text{End}_{HS}(H)$*

such that the subspaces $H_\sigma \subset H \otimes_{\mathbb{Q}} \mathbb{C}$ are defined over $\overline{\mathbb{Q}}$ for the de Rham structure.

Conjecture 1 (Gross-Deligne). *Let (H, F, ι) be a triple satisfying hypothesis H. Then the following relation holds for all $\lambda \in (\mathbb{Z}/d)^\times$:*

$$(7) \quad P_\lambda(H) \sim_{\overline{\mathbb{Q}}^\times} \prod_{a=1}^{d-1} \Gamma(1 - \frac{a}{d})^{\varepsilon(a/\lambda)}.$$

Remark 1 (Koblitz-Ogus). Although the function ε is not uniquely determined by p , the conjecture is independent of the choice of ε , thanks to the following result: if ε satisfies $\sum \varepsilon(a) \langle a\lambda \rangle = 0$ for all $\lambda \in (\mathbb{Z}/d)^\times$, then all $\prod \Gamma(1 - \frac{a}{d})^{\varepsilon(a/\lambda)}$ are algebraic [Del79, App.].

0.3.1. When H is the first Betti cohomology $H_B^1(A)$ of a CM abelian variety (A, F, ι) , the periods $P_\sigma(H)$ of Definition 1 agree with the integrals $P_\sigma(A, F, \iota)$ introduced in (2). In the case of elliptic curves, it is a funny exercise to combine the class number formula of F , as in [Gro78, p. 206], with the distribution property of the gamma function to show that the Gross-Deligne conjecture and the expression (3) are equivalent. In general, an argument along the same lines of [Col93, Prop. III.1.2], allows to deduce the statement from Anderson's theorem [And82, Thm. 2.1].

Theorem 1 (Anderson). *Let (A, F, ι) be an abelian variety with complex multiplication. If F is abelian, the Gross-Deligne conjecture holds for $H_B^1(A)$.*

0.4. **Main results.** For motivic Hodge structures satisfying Hypothesis H, Maillot and Rössler conjectured an alternating absolute value version of (7), and they proved it in some significant cases which constitute the main evidence for the Gross-Deligne conjecture outside abelian varieties. In particular, smooth, projective varieties with finite order automorphisms provide a great supply of such Hodge structures.

0.4.1. *The setting.* Fix an integer $d \geq 2$ and a primitive d -th root of unity ζ . Let X be a smooth, projective variety, of dimension n , over $\overline{\mathbb{Q}}$. Consider the Betti cohomology $H_B^j(X) := H^j(X_{\mathbb{C}}^{\text{an}}, \mathbb{Q})$ in some fixed degree j . Assume also that X is endowed with an

automorphism g acting with order d on $H_B^j(X)$. Then the map $\zeta \mapsto g^*$ induces an embedding of $\mathbb{Q}(\zeta)$ into $\text{End}_{HS}(H_B^j(X))$, whence a decomposition $H_B^j(X) \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{\lambda \in \mathbb{Z}/d} H_{\lambda}$. Note that the sum of primitive eigenspaces $\bigoplus_{\lambda \in (\mathbb{Z}/d)^{\times}} H_{\lambda}$ descends to a rational sub-Hodge structure $H \subset H_B^j(X)$, which is a vector space over $\mathbb{Q}(\zeta)$. Put $r = \dim_{\mathbb{Q}(\zeta)} H$ and consider the r -th exterior power of the $\mathbb{Q}(\zeta)$ -vector space H , regarded as a \mathbb{Q} -vector space. This yields a pure rational structure of weight jr and dimension $[\mathbb{Q}(\zeta) : \mathbb{Q}]$.

Definition 2 ($\det_{\mathbb{Q}(\zeta)} H_B^j(X)$). We will denote by $\det_{\mathbb{Q}(\zeta)} H_B^j(X)$ this Hodge structure.

The Künneth formula realizes $\det_{\mathbb{Q}(\zeta)} H_B^j(X)$ as a sub-Hodge structure of $H_B^{jr}(X^r)$, and one can show, as in [MR04, Lemma 2.1], that $\det_{\mathbb{Q}(\zeta)} H_B^j(X)$ satisfies Hypothesis H. Moreover, the associated function $p : (\mathbb{Z}/d)^{\times} \rightarrow \mathbb{Z}$ is given by

$$p(\lambda) = \sum_{p+q=j} p h_{\lambda}^{p,q}(X),$$

where $h_{\lambda}^{p,q}(X)$ is the dimension of the subspace of $H^q(X, \Omega_X^p)$ where g^* acts by ζ^{λ} .

Remark 2. In this case, the period $P_{\lambda}(\det_{\mathbb{Q}(\zeta)} H_B^j(X))$ admits a very concrete description: it is the determinant, with respect to $\overline{\mathbb{Q}}$ and $\mathbb{Q}(\zeta)$ -basis, of the restriction of the comparison isomorphism to the subspaces where g^* acts by the eigenvalue ζ^{λ} .

0.4.2. The main result of this paper is the following alternating version of the Gross-Deligne conjecture for the Hodge structures $\det_{\mathbb{Q}(\zeta)} H_B^j(X)$:

Theorem A. *Let X be a smooth, projective variety over $\overline{\mathbb{Q}}$. Assume that X is acted upon by an automorphism g of order d , and let $\gamma : \mathbb{Z}/d \rightarrow \mathbb{Q}$ be any function satisfying*

$$\frac{1}{d} \sum_{a \in \mathbb{Z}/d} \gamma(a) \langle au \rangle = \sum_{j=0}^{2n} (-1)^j \sum_{p+q=j} p h_{\lambda}^{p,q}(X)$$

for all $\lambda \in (\mathbb{Z}/d)^{\times}$. Then the following relation holds for all $\lambda \in (\mathbb{Z}/d)^{\times}$:

$$(8) \quad \prod_{j=0}^{2n} P_{\lambda}(\det_{\mathbb{Q}(\zeta)} H_B^j(X))^{(-1)^j} \sim_{\overline{\mathbb{Q}}^{\times}} \prod_{a=1}^{d-1} \Gamma(1 - \frac{a}{d})^{\gamma(a/\lambda)}.$$

Remark 3. To be completely precise, the construction of $\det_{\mathbb{Q}(\zeta)} H_B^j(X)$ only makes sense when g^* acts on $H_B^j(X)$ with order d , as we assumed before. As a matter of convenience, we will just set $P_{\lambda}(\det_{\mathbb{Q}(\zeta)} H_B^j(X)) = 1$ if this is not the case.

Remark 4. When d is prime, the identity in $\mathbb{R}^{\times} / (\mathbb{R} \cap \overline{\mathbb{Q}})^{\times}$ deduced from (8) by taking absolute value was already proved by Maillot and Rössler [MR04, §4], as a consequence of their more general result [MR04, Thm. 2].

0.4.3. In some interesting cases, theorem A suffices to prove the original Gross-Deligne conjecture for the Hodge structures under consideration. For instance:

Corollary 1. *If X is a smooth, complete intersection, of dimension n , over $\overline{\mathbb{Q}}$, together with an automorphism of order $d \geq 2$, the Gross-Deligne conjecture holds for $\det_{\mathbb{Q}(\zeta)} H_B^n(X)$.*

Proof. Let $\iota : X \hookrightarrow \mathbb{P}^N$ be a projective embedding of X . By the Lefschetz hyperplane theorem, $\iota^* : H^j(\mathbb{P}^N) \rightarrow H^j(X)$ is an isomorphism for all $j \neq n$. Since such a non-zero $H^j(X)$ is one-dimensional, g^* acts by multiplication by a d -th root of unity, which can only be ± 1 because integral cohomology is preserved. If $d \geq 3$, there are no primitive

eigenvalues. When $d = 2$, g^* may actually act as -1 in a few exceptional cases. But then $P_{-1}(H^j(X)) = (2\pi i)^{j/2}$, $h_{-1}^{j/2, j/2} = 1$ and the identity (7) amounts to $\Gamma(1/2) = \sqrt{\pi}$. At any rate, the conjecture holds for all $j \neq n$, so the statement follows from Theorem A. \square

0.4.4. Periods of cyclic covers. Remarkably, the proof of theorem A reduces by geometric arguments to the computation of periods of cyclic covers ramified along normal crossings divisors. In the simplest case of a curve, this is achieved by looking at the quotient $X/\langle g \rangle$, which remains smooth. In general, before taking the quotient one has to perform a sequence of equivariant blow-ups, much in the same vein as the ones used to study local systems on the complement of hyperplane arrangements [ESV92].

Let Z be a smooth, projective variety over $\overline{\mathbb{Q}}$ and let $D = \sum a_i D_i$ be an effective normal crossings divisor on Z . Assume that there is an invertible sheaf \mathcal{L} on Z and an integer $d \geq 2$ such that \mathcal{L}^d has a global section whose zero divisor is D . To these data, one associates a d -fold covering $\pi : Y \rightarrow Z$ ramified at D , with Galois group $\mu_d(\mathbb{C})$. Although Y need not be smooth, the Hodge structures $H_B^j(Y)$ fulfil Hypothesis H (see §2.1 below).

Theorem B. *Let $\gamma : \mathbb{Z}/d \rightarrow \mathbb{Q}$ be any function satisfying*

$$(9) \quad \frac{1}{d} \sum_{a \in \mathbb{Z}/d} \gamma(a) \langle a\lambda \rangle = \sum_{j=0}^{2n} (-1)^j \sum_{p+q=j} p h_{\lambda}^{p,q}(Y)$$

for all $\lambda \in (\mathbb{Z}/d)^{\times}$. Then the following relation holds for all $\lambda \in (\mathbb{Z}/d)^{\times}$:

$$(10) \quad \prod_{j=0}^{2n} P_{\lambda}(\det_{\mathbb{Q}(\zeta)} H_B^j(Y))^{(-1)^j} \sim_{\overline{\mathbb{Q}}}^{\times} \prod_{a=1}^{d-1} \Gamma(1 - \frac{a}{d})^{\gamma(a/\lambda)}.$$

0.4.5. Idea of the proof. Following an idea of Bloch [Blo05], to prove Theorem B, one first express $P_{\lambda}(\det_{\mathbb{Q}(\zeta)} H_B^j(Y))$ as the “determinants of periods” of a rank one logarithmic connection appearing in the decomposition of $\pi_* \mathcal{O}_Y$ according to the action of $\mu_d(\mathbb{C})$. By the Saito-Terasoma theorem, this determinant is equal, up to an algebraic factor, to the product of an explicit power of $2\pi i$ and a suitable product of gamma values at the residues of the connection along the different D_i . Since the monodromy is quasi-unipotent and the connection is trivialized by a degree d cover, these are rational numbers of denominator d . Taking into account the properties of the gamma function, this gives a relation like (10). Finally, to see that the exponents actually satisfy (9), one uses the Hirzebruch-Riemann-Roch theorem.

0.5. Outline. Section §1 contains a detailed presentation, as self-contained as possible, of the statement of the Saito-Terasoma theorem. We also give a small complement needed for the proof of Theorem B: the explicit computation of the periods of the unit object, whose proof is the object of Appendix A. This might be of independent interest, as it is my belief that the Saito-Terasoma theorem will find other applications in the future. Section §2 is devoted to the proof of theorem B, which is accomplished in §2.2, after some preliminaries on cyclic covers à la Esnault-Viehweg; we also present a variant with compact support in §2.3. Finally, in Section §3 we construct an equivariant tower of blow-ups of X which allows to deduce Theorem A from the results of Section §2.

Conventions. Most of the notations are standard. For the sake of precision, let us mention the following: (1) By an *algebraic variety* over some field k , we mean an integral separated k -scheme of finite type. (2) Given an algebraic variety X over a subfield k of \mathbb{C} , we will denote by $X_{\mathbb{C}}^{\text{an}}$ the complex manifold associated to the scalar extension $X \times_k \mathbb{C}$.

(3) A divisor D in X is said to have *normal crossings* if all its irreducible components are smooth and the intersection of any m of them has codimension m in X .

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1. THE SAITO-TERASOMA THEOREM

Throughout this section, k and F will denote subfields of \mathbb{C} , and U a smooth and quasi-projective variety over k .

1.1. The category $M_{k,F}(U)$. Following [ST97, Def. 2, p. 872], we shall consider the category $M_{k,F}(U)$ whose objects are triples $\mathcal{M} = ((\mathcal{E}, \nabla), V, \rho)$, where

- \mathcal{E} is a locally free \mathcal{O}_U -module of finite rank (i.e. a vector bundle) endowed with an integrable connection $\nabla : \mathcal{E} \rightarrow \mathcal{E} \otimes_{\mathcal{O}_U} \Omega_U^1$ with regular singularities,
- V is a local system of F -vector spaces on $U_{\mathbb{C}}^{\text{an}}$,
- $\rho : \ker(\nabla^{\text{an}}) \xrightarrow{\sim} V \otimes_F \mathbb{C}$ is an isomorphism of complex local systems on $U_{\mathbb{C}}^{\text{an}}$.

Given such an \mathcal{M} , we will call (\mathcal{E}, ∇) (resp. V) the de Rham (resp. Betti) component, and ρ the comparison isomorphism. We will write $\text{rk}(\mathcal{M})$ for the rank of \mathcal{E} .

Using resolution of singularities, one can find a smooth projective variety X over k containing U as the complement of a normal crossings divisor D . Recall that a connection $\nabla : \mathcal{E} \rightarrow \mathcal{E} \otimes_{\mathcal{O}_U} \Omega_U^1$ has *regular singularities* if there exists a coherent \mathcal{O}_X -sheaf \mathcal{E}_X and a logarithmic connection $\nabla_X : \mathcal{E}_X \rightarrow \mathcal{E}_X \otimes_{\mathcal{O}_X} \Omega_X^1(\log D)$ prolonging \mathcal{E} and ∇ respectively. Such an $(\mathcal{E}_X, \nabla_X)$ will be called a (logarithmic) extension of (\mathcal{E}, ∇) . We refer the reader to [Del70] or [Kat71] for basic facts concerning regular singular connections.

Example 1 (unit object). The trivial connection (\mathcal{O}_U, d) has regular singularities and gives rise to the so-called “unit object” $\mathbf{1} = ((\mathcal{O}_U, d), F, \ker(d^{\text{an}}) \simeq \mathbb{C} \simeq F \otimes_F \mathbb{C})$ in $M_{k,F}(U)$.

Example 2 (dual object). Given a vector bundle \mathcal{E} , let \mathcal{E}^\vee denote its dual, together with the pairing $(\cdot, \cdot) : \mathcal{E} \otimes_{\mathcal{O}_U} \mathcal{E}^\vee \rightarrow \mathcal{O}_U$. The requirement $d(e, f^\vee) = (\nabla(e), f^\vee) + (e, \nabla^\vee(f^\vee))$ uniquely determines a connection $(\mathcal{E}^\vee, \nabla^\vee)$, which has regular singularities if (\mathcal{E}, ∇) does. Since $\ker(\nabla^{\vee, \text{an}})$ and $\ker(\nabla^{\text{an}})^\vee$ are canonically isomorphic, each object $\mathcal{M} = ((\mathcal{E}, \nabla), V, \rho)$ has an associated dual $\mathcal{M}^\vee = ((\mathcal{E}^\vee, \nabla^\vee), V^\vee, \rho^\vee)$ in $M_{k,F}(U)$.

Example 3 (determinant). To each $\mathcal{M} = ((\mathcal{E}, \nabla), V, \rho)$ in $M_{k,F}(U)$, it is attached the rank one object $\det(\mathcal{M}) = ((\det \mathcal{E}, \text{tr}(\nabla)), \det V, \det \rho)$. If $r = \text{rk}(\mathcal{M})$, we endowed the line bundle $\det \mathcal{E} := \Lambda_{\mathcal{O}_U}^r \mathcal{E}$ with the connection $\text{tr}(\nabla)(e_1 \wedge \dots \wedge e_r) = \sum e_1 \wedge \dots \wedge \nabla(e_i) \wedge \dots \wedge e_r$. One easily verifies that $(\det \mathcal{E}, \text{tr}(\nabla))$ has regular singularities and that $\ker(\text{tr}(\nabla)^{\text{an}}) \simeq \Lambda^r \ker(\nabla^{\text{an}})$, so ρ induces an isomorphism $\det \rho : \ker(\text{tr}(\nabla)^{\text{an}}) \rightarrow \det V \otimes_F \mathbb{C}$.

1.1.1. Rank one objects. Let $P_{k,F}(U)$ be the subcategory of $M_{k,F}(U)$ consisting of rank one objects and isomorphisms, and $\text{MPic}_{k,F}(U)$ the set of isomorphism classes. The tensor product of connections $(\mathcal{E}_1, \nabla_1) \otimes (\mathcal{E}_2, \nabla_2) = (\mathcal{E}_1 \otimes_{\mathcal{O}_U} \mathcal{E}_2, \nabla_1 \otimes \text{id}_{\mathcal{E}_2} + \text{id}_{\mathcal{E}_1} \otimes \nabla_2)$ induces a group structure on $\text{MPic}_{k,F}(U)$, with identity the class of $\mathbf{1}$ and inverse given by the dual objects. Note that the determinant gives a functor $\det : M_{k,F}(U) \rightarrow P_{k,F}(U)$.

1.1.2. For a finite extension L of k , objects in $M_{k,F}(\text{Spec}(L))$ are just triples $(M_{\text{dR}}, M_B, \rho)$ made of an L -vector space M_{dR} , an F -vector space M_B and an isomorphism $\rho : M_{\text{dR}} \otimes_k \mathbb{C} \xrightarrow{\sim} M_B \otimes_F \mathbb{C}$. Via the usual identification $L \otimes_k \mathbb{C} \simeq \mathbb{C}^{\text{Hom}_k(L, \mathbb{C})}$, one can think of ρ as the data of $\rho_\sigma : M_{\text{dR}} \otimes_{L, \sigma} \mathbb{C} \xrightarrow{\sim} M_B \otimes_F \mathbb{C}$, as σ runs through $\text{Hom}_k(L, \mathbb{C})$. If \mathcal{M} has rank one, let e and v be rational basis of M_{dR} and M_B and define $\alpha_\sigma \in \mathbb{C}^\times$ by $\rho_\sigma(e \otimes_{L, \sigma} 1_{\mathbb{C}}) = \alpha_\sigma v \otimes_F 1_{\mathbb{C}}$. Then the map $[\mathcal{M}] \mapsto (\alpha_\sigma)_\sigma$ induces an isomorphisms of groups

$$(11) \quad \text{MPic}_{k,F}(\text{Spec}(L)) \simeq k^\times \backslash (\mathbb{C}^\times)^{\text{Hom}_k(L, \mathbb{C})} / (F^\times)^{\text{Hom}_k(L, \mathbb{C})},$$

as well as a norm functor

$$(12) \quad N_{L/k} : \text{MPic}_{k,F}(\text{Spec}(L)) \longrightarrow \text{MPic}_{k,F}(\text{Spec}(k)).$$

Example 4 (fiber at a closed point). Let $\mathcal{M} = ((\mathcal{E}, \nabla), V, \rho)$ be an object in $\text{MPic}_{k,F}(U)$, and x a closed point of U , with residue field $k(x)$. Denote by $\mathcal{E}_x := \mathcal{E} \otimes_{\mathcal{O}_{U,x}} k(x)$ the fiber at x . For each $\sigma : k(x) \hookrightarrow \mathbb{C}$, there is a canonical isomorphism $\mathcal{E}_x \otimes_{k(x), \sigma} \mathbb{C} \simeq \ker(\nabla^{\text{an}})_x$, hence an object $\mathcal{M}|_x = (\mathcal{E}_x, V_x, \rho_x)$ in $P_{k,F}(\text{Spec}(k(x)))$. In concrete terms, if φ is an analytic horizontal section of ∇ on a sufficiently small neighborhood of x and v is a local basis of V , then $[\mathcal{M}|_x]$ is given by $(\varphi(\sigma(x)) / \rho(v \otimes_F 1_{\mathbb{C}}))_\sigma$. More generally, if \mathcal{M} has arbitrary rank, one can make the same construction on $\det(\mathcal{M})$. Taking the norm down to k , one gets a class in $\text{MPic}_{k,F}(\text{Spec}(k))$.

1.2. **Periods of connections.** An integrable connection $\nabla : \mathcal{E} \rightarrow \mathcal{E} \otimes_{\mathcal{O}_U} \Omega_U^1$ canonically extends to a complex of sheaves $(\mathcal{E} \otimes_{\mathcal{O}_U} \Omega_U^\bullet, \nabla)$ on U which will be denoted by $DR(\mathcal{E}, \nabla)$. By analytification, one deduces a complex of sheaves $DR(\mathcal{E}^{\text{an}}, \nabla^{\text{an}})$ on $U_{\mathbb{C}}^{\text{an}}$ and canonical morphisms of complex vector spaces

$$(13) \quad \mathbb{H}^\bullet(U, DR(\mathcal{E}, \nabla)) \otimes_k \mathbb{C} \longrightarrow \mathbb{H}^\bullet(U_{\mathbb{C}}^{\text{an}}, DR(\mathcal{E}^{\text{an}}, \nabla^{\text{an}})).$$

Under the assumption that (\mathcal{E}, ∇) has regular singularities, Deligne proved that this map is an isomorphism [Del70, Ch. II, Thm. 6.2]. Besides, $DR(\mathcal{E}^{\text{an}}, \nabla^{\text{an}})$ is a resolution of the local system of horizontal sections $\ker(\nabla^{\text{an}})$ [Del70, Ch. I, Prop. 2.19], whence:

$$(14) \quad H^\bullet(U_{\mathbb{C}}^{\text{an}}, \ker(\nabla^{\text{an}})) \xrightarrow{\sim} \mathbb{H}^\bullet(U_{\mathbb{C}}^{\text{an}}, DR(\mathcal{E}^{\text{an}}, \nabla^{\text{an}})).$$

If, in addition, an F -structure on $\ker(\nabla^{\text{an}})$ is given, that is, an F -local system V on $U_{\mathbb{C}}^{\text{an}}$ and an isomorphism $p : \ker(\nabla^{\text{an}}) \xrightarrow{\sim} V \otimes_F \mathbb{C}$, one gets isomorphisms

$$(15) \quad H^\bullet(\rho) : \mathbb{H}^\bullet(U, DR(\mathcal{E}, \nabla)) \otimes_k \mathbb{C} \xrightarrow{\sim} H^\bullet(U_{\mathbb{C}}^{\text{an}}, V) \otimes_F \mathbb{C}$$

from (13), (14) and the map induced by ρ on cohomology.

Definition 3 (determinant of periods). Let $\mathcal{M} = ((\mathcal{E}, \nabla), V, \rho)$ be an object in $M_{k,F}(U)$. The determinant of periods of \mathcal{M} is the following rank one object in $M_{k,F}(\text{Spec}(k))$:

$$\text{per}(\mathcal{M}) = \left(\bigotimes_{j=0}^{2n} \det \mathbb{H}^j(U, DR(\mathcal{E}, \nabla))^{(-1)^j}, \bigotimes_{j=0}^{2n} \det H^j(U_{\mathbb{C}}^{\text{an}}, V)^{(-1)^j}, \bigotimes_{j=0}^{2n} \det H^j(\rho)^{(-1)^j} \right).$$

We shall often regard $\text{per}(\mathcal{M})$ as an element in $k^\times \backslash \mathbb{C}^\times / F^\times$ through the isomorphism (11).

1.2.1. *Periods of the unit object.* When \mathcal{M} is the unit object in $M_{k,\mathbb{Q}}(U)$, the above construction gives back Grothendieck's comparison isomorphism [Gr66]

$$H_{\text{dR}}^\bullet(U/k) \otimes_k \mathbb{C} \xrightarrow{\sim} H^\bullet(U_{\mathbb{C}}^{\text{an}}, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}.$$

In this case, one can actually prove that $\text{per}(\mathbf{1})$ is always a power of $2\pi i$, up to some factor of order at most two over k , a result generalizing Legendre's relation for elliptic

curves. Clearly, the same follows for $\text{per}(\mathbf{1})$ in $M_{k,F}(U)$. In order not to lose the thread of the exposition, the proof and a few comments will be deferred to Appendix A.

Proposition 1. *The following relation holds in $\mathbb{C}^\times / k^\times$:*

$$\text{per}(\mathbf{1})^2 = (2\pi i)^{2\sum_{p=0}^n (-1)^p \text{ph}^q(X, \Omega_X(\log D))}.$$

1.2.2. The Saito-Terasoma theorem gives an expression for $\text{per}(\mathcal{M})$ in $k^\times \backslash \mathbb{C}^\times / F^\times$. It involves three terms: the periods of the unit object $\text{per}(\mathbf{1})$, a product of gamma values depending only on the de Rham component of \mathcal{M} , and the pairing of $\det(\mathcal{M})$ with a canonical relative zero-cycle. Let us introduce them.

1.3. **The gamma factor.** Let $(\mathcal{E}_X, \nabla_X)$ be a logarithmic extension of (\mathcal{E}, ∇) to X . Given an irreducible component D_i of D , we shall denote by $\mathcal{E}_{X|D_i} := \mathcal{E}_X \otimes_{\mathcal{O}_X} \mathcal{O}_{D_i}$ the restriction of \mathcal{E}_X to D_i and by $k_i := \Gamma(D_i, \mathcal{O}_{D_i})$ the constant field of D_i , a finite extension of k . For each D_i , the composition of ∇_X and the Poincaré residue

$$\mathcal{E}_X \xrightarrow{\nabla_X} \mathcal{E}_X \otimes_{\mathcal{O}_X} \Omega_X^1(\log D) \xrightarrow{\text{id} \otimes \text{Res}_{D_i}} \mathcal{E}_X \otimes_{\mathcal{O}_X} \mathcal{O}_{D_i} = \mathcal{E}_{X|D_i}$$

induces an $(\mathcal{O}_{D_i}$ -linear) endomorphism $\text{Res}_{D_i} \nabla_X \in \text{End}(\mathcal{E}_{X|D_i})$, called the *residue of the connection along D_i* . Consider its characteristic polynomial

$$\Phi_{\mathcal{E}_X, i}(T) = \det(T - \text{Res}_{D_i} \nabla_X) \in k_i[T].$$

Classically, the roots of $\Phi_{\mathcal{E}_X, i}(T)$ are called *exponents* of $(\mathcal{E}_X, \nabla_X)$ at D_i . Following the terminology of [ST97, Def. 1, p. 870], we will say that an extension is *small* if none of the exponents at none of the components D_i is an integer $n \leq 0$ ⁴.

Remark 5. The exponents depend on the extension, whereas their classes modulo \mathbb{Z} are determined by (\mathcal{E}, ∇) . Indeed, if α is an exponent of $(\mathcal{E}_X, \nabla_X)$ at D_i , then $\exp(-2\pi i \alpha)$ is an eigenvalue of the local monodromy of $\ker(\nabla^{\text{an}})$ around D_i [Del70, Ch. II, Prop. 3.11].

If $P \in k[T]$ is a polynomial such that $P(n) \neq 0$ for all integers, we put

$$\Gamma(P) := \prod_{\substack{\alpha \in \mathbb{C} \\ P(\alpha) = 0}} \Gamma(\alpha) \in \mathbb{C}^\times.$$

With this notation, the gamma factor of a small extension $(\mathcal{E}_X, \nabla_X)$ of (\mathcal{E}, ∇) is

$$(16) \quad \Gamma(\nabla_X : \mathcal{E}_X) = \prod_{i \in I} \Gamma(N_{k_i/k} \Phi_{\mathcal{E}_X, i})^{\chi(D_i^\circ)} \in \mathbb{C}^\times,$$

where $D_i^\circ := D_i - \bigcup_{i \neq j} (D_i \cap D_j)$ and $N_{k_i/k} : k_i(T) \rightarrow k(T)$ stands for the norm.

Using remark 5 and the functional equation $\Gamma(s+1) = s\Gamma(s)$, one can show that the class of $\Gamma(\nabla_X : \mathcal{E}_X)$ modulo k^\times depends only on (\mathcal{E}, ∇) and not on the small extension chosen to define it [ST97, Lemma 1.8], hence the following definition:

Definition 4 (gamma factor). The gamma factor of an object $\mathcal{M} = ((\mathcal{E}, \nabla), V, \rho)$ in $M_{k,F}(U)$ is the class $\Gamma(\nabla : \mathcal{E}) \in \mathbb{C}^\times / k^\times$ of $\Gamma(\mathcal{E}_X : \nabla_X)$ for any small extension $(\mathcal{E}_X, \nabla_X)$ ⁵.

We shall need later the following alternative expression for the exponents in $\Gamma(\nabla : \mathcal{E})$:

$$\text{Lemma 1. } \chi(D_i^\circ) = \int_{D_i} c_{n-1}(\Omega_X^1(\log D)|_{D_i}).$$

⁴Small or big extensions always exist since, starting from any extension $(\mathcal{E}_X, \nabla_X)$, one can define a logarithmic connection on $\mathcal{E}_X \otimes_{\mathcal{O}_X} (\sum \mu_j D_j)$ whose residue along D_i is $\text{Res}_{D_i} \nabla_X - \mu_i \cdot \text{id}_{\mathcal{O}_{D_i}}$ [EV97, Lemma 2.7].

⁵Saito and Terasoma use the notation $\Gamma(\nabla : \mathcal{M})$. I will prefer to emphasize that the gamma factor depends only on the de Rham component of \mathcal{M} .

1.4. The pairing with the canonical relative zero-cycle. Let $\mathcal{K}_{n,X}$ be the Zariski sheafification of Quillen's K-theory $V \mapsto K_n(V)$ [Qui72, §7]. Consider the complex

$$\mathcal{K}_n(X \bmod D) := [\mathcal{K}_{n,X} \longrightarrow \bigoplus_{i \in I} \iota_{i*} \mathcal{K}_{n,D_i}],$$

where $\mathcal{K}_{n,X}$ sits in degree zero and $\iota_i : D_i \hookrightarrow X$ denotes the inclusion.

Definition 5. The relative Chow group of zero-cycles is

$$\mathrm{CH}^n(X \bmod D) := \mathbb{H}^n(X, \mathcal{K}_n(X \bmod D)).$$

In view of the Bloch-Quillen formula $\mathrm{CH}^p(X) \simeq \mathbb{H}^p(X, \mathcal{K}_{p,X})$ [Qui72, Thm. 5.19], the exact sequence of complexes $0 \rightarrow \bigoplus_{i \in I} \iota_{i*} \mathcal{K}_{n,D_i}[-1] \rightarrow \mathcal{K}_n(X \bmod D) \rightarrow \mathcal{K}_{n,X} \rightarrow 0$ induces a long exact sequence of groups

$$(17) \quad \cdots \longrightarrow \bigoplus_{i \in I} H^{n-1}(D_i, \mathcal{K}_{n,D_i}) \longrightarrow \mathrm{CH}^n(X \bmod D) \longrightarrow \mathrm{CH}^n(X) \longrightarrow 0.$$

Example 5. When X is a curve, it follows from the sequence (17) that

$$\mathrm{CH}^1(X \bmod D) \simeq \left(\bigoplus_{x \in |U|} \mathbb{Z} \right) / \{ \mathrm{div}(f) \mid f \in k(X)^\times, f \equiv 1 \pmod{D} \},$$

so one gets back the divisor class group with modulus.

In general, Saito and Terasoma prove that $\mathrm{CH}^1(X \bmod D)$ is generated by the classes of closed points of U (in fact, a dense open subscheme of U suffices) [ST97, Cor. p. 888]. Thus, to define a pairing with $\mathrm{Mpic}_{k,F}(U)$, it suffices to define it on each such $[x]$:

Definition 6 (pairing with the relative Chow group). The pairing

$$(\cdot) : \mathrm{Mpic}_{k,F}(U) \times \mathrm{CH}^n(X \bmod D) \rightarrow \mathrm{Mpic}_{k,F}(\mathrm{Spec}(k))$$

is defined by the requirement that $(\mathcal{M}, [x]) = N_{k(x)/k}(\mathcal{M}|_x)$ for all closed points x of U .

1.4.1. By a construction due to Saito [Sai93, §1], given a vector bundle \mathcal{F} of rank n on X and a partial trivialization along D , there is a lifting of the usual top Chern class of \mathcal{F} to $\mathrm{CH}^n(X \bmod D)$ called the relative top Chern class. Here *partial trivialization* means a family $r = (r_i)_{i \in I}$ of surjective morphisms $r_i : \mathcal{F}|_{D_i} \rightarrow \mathcal{O}_{D_i}$. The definition goes through local hypercohomology, for which we refer the reader e.g. to [Dim04, §2.4]. Letting $V = \mathbf{V}(\mathcal{F})$ denote the geometric vector bundle associated to \mathcal{F} , r_i induces a morphism $r_i : V|_{D_i} \rightarrow \mathbb{A}_{D_i}^1$. Define $\Delta_i = r_i^{-1}(1)$, where $1 \in \mathbb{A}_{D_i}^1$ denotes the 1-section. Then $\Delta = (\Delta_i)_{i \in I}$ is a family of regular subschemes of V , disjoint from the zero section $\{0\}$ of V , and the Gysin map associated to $V \rightarrow X$ induces an isomorphism

$$(18) \quad H^0(X, \mathbb{Z}) \xrightarrow{\sim} H_{\{0\}}^n(V, K_{n,V}) \simeq \mathbb{H}_{\{0\}}^n(V, \mathcal{K}_n(V \bmod \Delta))$$

by the relative purity of K-theory. On the other hand, since both $V \rightarrow X$ and $\Delta_i \rightarrow D_i$ are affine bundles, by the homotopy property of K-theory [Qui72, Prop. 4.1], the complexes $\mathcal{K}_n(X \bmod D)$ and $\mathcal{K}_n(V \bmod \Delta)$ are isomorphic, whence

$$(19) \quad \mathbb{H}^n(V, \mathcal{K}_n(V \bmod \Delta)) \xrightarrow{\sim} \mathbb{H}^n(X, \mathcal{K}_n(X \bmod D)) = \mathrm{CH}^n(X \bmod D).$$

The composition of (18), the natural map $\mathbb{H}_{\{0\}}^n(V, \mathcal{K}_n(V \bmod \Delta)) \rightarrow \mathbb{H}^n(V, \mathcal{K}_n(V \bmod \Delta))$ and (19) induces a morphism $H^0(X, \mathbb{Z}) \rightarrow \mathrm{CH}^n(X \bmod D)$, and one defines $c_n(\mathcal{E}_X, r) \in \mathrm{CH}^n(X \bmod D)$ to be the image of $1 \in H^0(X, \mathbb{Z})$.

Definition 7 (relative canonical cycle). Let $\mathrm{Res} = (\mathrm{Res}_{D_i})_{i \in I}$ be the partial trivialization of $\Omega_X^1(\log D)$ given by the Poincaré residues. The relative canonical cycle is

$$c_{X \bmod D} := (-1)^n c_n(\Omega_X^1(\log D), \mathrm{Res}).$$

1.5. Statement of the theorem. We finally come to the main result of [ST97]:

Theorem 2 (Saito-Terasoma). *Let $\mathcal{M} = ((\mathcal{E}, \nabla), V, \rho)$ be an object in $M_{k,F}(U)$. The following relation holds in $k^\times \backslash \mathbb{C}^\times / F^\times$:*

$$\text{per}(\mathcal{M}) = \text{per}(\mathbf{1})^{\text{rk}(\mathcal{M})} \cdot (\det(\mathcal{M}), c_{X \bmod D}) \cdot \Gamma(\nabla^\vee : \mathcal{E}^\vee).$$

2. PERIODS OF CYCLIC COVERS

2.1. Preliminaries on cyclic covers à la Esnault-Viehweg. Let Z be a smooth projective variety over $\overline{\mathbb{Q}}$, and let $D = \sum a_i D_i$ be an effective normal crossings divisor on Z , with pairwise distinct irreducible components D_i indexed by a finite set I . Assume that there exists an invertible sheaf \mathcal{L} on Z such that $\mathcal{O}_Z(D) \simeq \mathcal{L}^d$ for some integer $d \geq 2$; that is, \mathcal{L}^d admits a global section s whose zero divisor is D . Recall from [EV97, §3.5, 3.14] that, out of these data, one can form a d -fold cyclic cover $\pi : Y \rightarrow Z$ ramified at D . The quickest way to construct it is as follows: consider the geometric rank one vector bundles $\mathbf{V}(\mathcal{L}^{-1})$ and $\mathbf{V}(\mathcal{L}^{-d})$ associated to \mathcal{L}^{-1} and \mathcal{L}^{-d} respectively, and let $\tau : \mathbf{V}(\mathcal{L}^{-1}) \rightarrow \mathbf{V}(\mathcal{L}^{-d})$ denote the obvious map between them. Then s corresponds to a geometric section $\sigma : Z \rightarrow \mathbf{V}(\mathcal{L}^{-d})$, and one defines $\pi : Y \rightarrow Z$ as the normalization of $\tau^{-1}(\sigma(Z))$, equipped with the restriction of the projection $\mathbf{V}(\mathcal{L}^{-1}) \rightarrow Z$.

A local computation shows that π is étale over $Z - D$ and that the singular locus Σ of Y lies above the singular locus of D [EV97, §3.15]. Furthermore, Y has at most finite quotient singularities [EV97, §3.24]. It follows that any desingularization $\tilde{Y} \rightarrow Y$ induces an injective morphism of Hodge structures $H_B^j(Y) \hookrightarrow H_B^j(\tilde{Y})$ [Del74, Thm. 8.2.4], so $H_B^j(Y)$ is pure of weight j . On the de Rham side, Y comes with the complex of coherent sheaves $\tilde{\Omega}_Y^\bullet = i_* \Omega_{Y-\Sigma}^\bullet$, where $i : Y - \Sigma \hookrightarrow Y$ is the inclusion of the smooth locus. Letting $H_{dR}^j(Y)$ denote its hypercohomology in degree j , the spectral sequence $E_1^{p,q} = H^q(Y, \tilde{\Omega}_Y^p) \rightarrow H_{dR}^j(Y)$ degenerates at E_1 , and there is a canonical isomorphism

$$(20) \quad H_{dR}^j(Y) \otimes_{\overline{\mathbb{Q}}} \mathbb{C} \xrightarrow{\sim} H_B^j(Y) \otimes_{\mathbb{Q}} \mathbb{C},$$

which sends the filtration induced by E_1 to the Hodge filtration [Ste77, §1].

2.1.1. The Galois group of the cover $G \simeq \mu_d(\mathbb{C})$ acts on $\pi_* \mathcal{O}_Y$. In order to analyze this action, it is useful to introduce –after Esnault and Viehweg– the invertible sheaves

$$(21) \quad \mathcal{L}^{(\lambda)} := \mathcal{L}^\lambda \otimes_{\mathcal{O}_Z} \mathcal{O}_Z(-\sum_{i \in I} [\frac{a_i \lambda}{d}] D_i),$$

where λ is an integer and $[\cdot]$ the integral part of a rational number. Note that $\mathcal{L}^{(\lambda)}$ depends only on the image of λ in \mathbb{Z}/d and that $\mathcal{L}^{(0)} = \mathcal{O}_Z$. Besides,

$$(22) \quad \mathcal{L}^{(\lambda)} \otimes_{\mathcal{O}_Z} \mathcal{L}^{(-\lambda)} \simeq \mathcal{O}_Z(-\sum_{i \in I} ([\frac{a_i \lambda}{d}] + [-\frac{a_i \lambda}{d}]) D_i) \simeq \mathcal{O}_Z(D^{(\lambda)}),$$

where the notation $D^{(\lambda)}$ stands for the *reduced* divisor with support the irreducible components D_i of D such that d does not divide $a_i \lambda$. Put also $U^{(\lambda)} := Z - D^{(\lambda)}$.

Lemma 2 (Esnault-Viehweg).

- (a) *Once a primitive d -th root of unity ζ is fixed, the subsheaf of $\pi_* \mathcal{O}_Y$ where G acts through the character $\chi(\zeta) = \zeta^\lambda$ is isomorphic to $\mathcal{L}^{(\lambda)^{-1}}$. Thus,*

$$\pi_* \mathcal{O}_Y = \bigoplus_{\lambda \in \mathbb{Z}/d} \mathcal{L}^{(\lambda)^{-1}}.$$

(b) Similarly, the decomposition of $\pi_* \tilde{\Omega}_Y^p$ into eigensheaves is given by

$$\pi_* \tilde{\Omega}_Y^p = \bigoplus_{\lambda \in \mathbb{Z}/d} \mathcal{L}^{(\lambda)^{-1}} \otimes_{\mathcal{O}_Z} \Omega_Z^p(\log D^{(\lambda)}).$$

Proof. See [EV97, Lemma 3.16] and [Ara14, Lemma 1.2]. \square

It follows that the exterior differential on Y induces logarithmic connections

$$\nabla^{(\lambda)} : \mathcal{L}^{(\lambda)^{-1}} \longrightarrow \mathcal{L}^{(\lambda)^{-1}} \otimes_{\mathcal{O}_Z} \Omega_Z^1(\log D^{(\lambda)})$$

such that the direct image of $\tilde{\Omega}_Y^\bullet$ splits into a direct sum of the associated de Rham complexes $DR(\mathcal{L}^{(\lambda)^{-1}}, \nabla^{(\lambda)})$. Locally, if t^{-1} is a generator of \mathcal{L} and $f = f_1^{a_1} \cdots f_r^{a_r}$ is an equation for D , one has $t^d = f$ and $\mathcal{L}^{(\lambda)^{-1}}$ is generated by $\sigma_\lambda = t^\lambda \cdot f_1^{-[a_1 \lambda/d]} \cdots f_r^{-[a_r \lambda/d]}$ [EV86, p.171]. Then a straightforward computation shows that⁶

$$\nabla^{(\lambda)}(\sigma_\lambda) = \sigma_\lambda \cdot \sum_{i=1}^r \left(\frac{a_i \lambda}{d} - \left\lfloor \frac{a_i \lambda}{d} \right\rfloor \right) \frac{df_i}{f_i}.$$

In particular, if $\langle \cdot \rangle$ denotes, as before, the representative between 0 and $d-1$:

$$(23) \quad \text{Res}_{D_i} \nabla^{(\lambda)} = \frac{\langle a_i \lambda \rangle}{d} \text{id}_{\mathcal{O}_{D_i}}.$$

2.1.2. The direct images of the constant sheaves $\mathbb{Q}(\zeta)$ and \mathbb{C} on $Y_{\mathbb{C}}^{\text{an}}$ also carry an action of G . For each $\lambda \in \mathbb{Z}/d$, let V_λ be the subsheaf of $\pi_* \mathbb{Q}(\zeta)$ where G acts through the character $\chi(\zeta) = \zeta^\lambda$. Then one has

$$\pi_* \mathbb{Q}(\zeta) = \bigoplus_{\lambda \in \mathbb{Z}/d} V_\lambda,$$

and $V_\lambda \otimes_{\mathbb{Q}(\zeta)} \mathbb{C}$ identifies with the corresponding eigensheaf of $\pi_* \mathbb{C}$. On $U^{(\lambda)}$ these are rank one local systems and, everything being compatible with the action of G , the associated vector bundle with connection is $(\mathcal{L}^{(\lambda)^{-1}}, \nabla^{(\lambda)})|_{U^{(\lambda)}}$. From this one gets an isomorphism $\rho_\lambda : \ker(\nabla^{(\lambda), \text{an}}) \simeq V_\lambda \otimes_{\mathbb{Q}(\zeta)} \mathbb{C}$, hence a rank one object

$$(24) \quad \mathcal{M}_\lambda = ((\mathcal{L}^{(\lambda)^{-1}}, \nabla^{(\lambda)})|_{U^{(\lambda)}}, V_\lambda, \rho_\lambda)$$

in the category $M_{\overline{\mathbb{Q}}, \mathbb{Q}(\zeta)}(U^{(\lambda)})$ of Saito and Terasoma.

Lemma 3.

(a) The Hodge structure $\det_{\mathbb{Q}(\zeta)} H_B^j(Y)$ satisfies Hypothesis H and the following identity holds in $\mathbb{C}^\times / \overline{\mathbb{Q}}^\times$ for all $\lambda \in (\mathbb{Z}/d)^\times$:

$$\text{per}(\mathcal{M}_\lambda) = \prod_{j=0}^{2n} P_\lambda(\det_{\mathbb{Q}(\mu_d)} H_B^j(Y))^{(-1)^j}.$$

(b) The relation $h_\lambda^{p,q}(Y) = h^q(Z, \mathcal{L}^{(\lambda)^{-1}} \otimes_{\mathcal{O}_Z} \Omega_Z^p(\log D^{(\lambda)}))$ holds for all $\lambda \in \mathbb{Z}/d$.

Proof. Since the morphism $\pi : Y \rightarrow Z$ is finite, one has

$$\begin{aligned} H_{dR}^j(Y) &= \mathbb{H}^j(Z, \pi_* \tilde{\Omega}_Y^\bullet) = \bigoplus_{\lambda \in \mathbb{Z}/d} \mathbb{H}^j(Z, DR(\mathcal{L}^{(\lambda)^{-1}}, \nabla^{(\lambda)})) \\ H_B^j(Y) \otimes_{\mathbb{Q}} \mathbb{Q}(\zeta) &= H^j(Z_{\mathbb{C}}^{\text{an}}, \pi_* \mathbb{Q}(\zeta)) = \bigoplus_{\lambda \in \mathbb{Z}/d} H^j(Z_{\mathbb{C}}^{\text{an}}, V_\lambda), \end{aligned}$$

⁶I hope the reader will forgive me for this momentaneous double use of the letter d .

and the comparison isomorphism (20) restricts to

$$\mathbb{H}^j(Z, DR(\mathcal{L}^{(\lambda)^{-1}}, \nabla^{(\lambda)})) \otimes_{\overline{\mathbb{Q}}} \mathbb{C} \xrightarrow{\sim} H^j(Z_{\mathbb{C}}^{\text{an}}, V_{\lambda}) \otimes_{\mathbb{Q}(\zeta)} \mathbb{C}.$$

It follows that the eigenspaces of $H_B^j(Y)$ are defined over $\overline{\mathbb{Q}}$, so Hypothesis H holds. Since the spectral sequence $H^q(Y, \tilde{\Omega}_Y^p) \Rightarrow H_{dR}^j(Y)$ is compatible with direct image and with the action of G , this also proves (b). For the second part of (a), it suffices to observe that $\nabla^{(\lambda)}$ has no integral residues along any of the components D_i , so

$$\mathbb{H}^j(Z, DR(\mathcal{L}^{(\lambda)^{-1}}, \nabla^{(\lambda)})) \simeq \mathbb{H}^j(U^{(\lambda)}, DR((\mathcal{L}^{(\lambda)^{-1}}, \nabla^{(\lambda)})|_{U^{(\lambda)}}))$$

and the same holds for $H^j(Z_{\mathbb{C}}^{\text{an}}, V_{\lambda}) \simeq H^j(U_{\mathbb{C}}^{(\lambda), \text{an}}, V_{\lambda|U^{(\lambda)}})$. \square

2.2. Proof of theorem B. Recall that we want to compute the alternating product

$$\text{per}(\mathcal{M}_{\lambda}) = \prod_{j=0}^{2n} P_{\lambda}(\det_{\mathbb{Q}(\zeta)} H_B^j(Y))^{(-1)^j} \in \mathbb{C}^{\times} / \overline{\mathbb{Q}}^{\times}$$

for all $\lambda \in (\mathbb{Z}/d)^{\times}$. Applying the Saito-Terasoma theorem to \mathcal{M}_{λ} , one gets

$$\text{per}(\mathcal{M}_{\lambda}) = \text{per}(\mathbf{1}) \cdot (\mathcal{M}_{\lambda}, c_{Z \bmod D^{(\lambda)}}) \cdot \Gamma(\nabla^{(-\lambda)} : \mathcal{L}^{(-\lambda)^{-1}})$$

since the objects \mathcal{M}_{λ} and $\mathcal{M}_{-\lambda}$ are dual to each other by (22). Observe that:

(a) Since we are working modulo $\overline{\mathbb{Q}}^{\times}$, Proposition 1 completely determines

$$\text{per}(\mathbf{1}) = (2\pi i)^{\sum_{k=0}^{2n} (-1)^k \sum_{p+q=j} ph^q(\Omega_Z^p(\log D^{(\lambda)}))}.$$

(b) By (23), the extension $\nabla^{(-\lambda)} : \mathcal{L}^{(-\lambda)^{-1}} \rightarrow \Omega_Z^1(\log D^{(\lambda)})$ has residues

$$\text{Res}_{D_i} \nabla^{(-\lambda)} = \frac{\langle -a_i \lambda \rangle}{d} = 1 - \frac{\langle a_i \lambda \rangle}{d},$$

so it is small and the gamma factor equals, by definition,

$$\Gamma(\nabla^{(-\lambda)} : \mathcal{L}^{(-\lambda)^{-1}}) = \prod_{i \in I} \Gamma(1 - \frac{\langle a_i \lambda \rangle}{d}) \chi^{(D_i^{\circ})}.$$

It remains to compute the pairing with the relative canonical zero-cycle:

Lemma 4. *The relation $(\mathcal{M}_{\lambda}, c_{Z \bmod D^{(\lambda)}}) = 1$ holds in $\mathbb{C}^{\times} / \overline{\mathbb{Q}}^{\times}$.*

Proof. Since $c_{Z \bmod D^{(\lambda)}}$ is a linear combination of classes of closed points x in U and one has $(\mathcal{M}_{\lambda}, [x]) = \mathcal{M}_{|x}$, it suffices to show that $\mathcal{M}_{|x} = 1$ in $\mathbb{C}^{\times} / \overline{\mathbb{Q}}^{\times}$ for all such x . But this follows directly from the description given in Example 4, since the base field is $\overline{\mathbb{Q}}$ and the $\mathbb{Q}(\zeta)$ -structure on $\ker(\nabla^{\text{an}})_x$ is induced by the natural inclusion $\mathbb{Q}(\zeta) \subset \mathbb{C}$. \square

Thus, putting everything together:

$$(25) \quad \text{per}(\mathcal{M}_{\lambda}) = (2\pi i)^{\sum_{k=0}^{2n} (-1)^k \sum_{p+q=j} ph^q(\Omega_Z^p(\log D^{(\lambda)}))} \cdot \prod_{i \in I} \Gamma(1 - \frac{\langle a_i \lambda \rangle}{d}) \chi^{(D_i^{\circ})}.$$

Now recall that the Gamma function satisfies the distribution property

$$\Gamma(s) = (2\pi)^{\frac{d-1}{2}} d^{s-\frac{1}{2}} \prod_{a=1}^{d-1} \Gamma(\frac{s+a}{d}),$$

which, making $s = 1$, gives $(2\pi)^{\frac{d-1}{2}} \sim_{\overline{\mathbb{Q}}^\times} \Gamma(\frac{1}{d}) \cdots \Gamma(\frac{d-1}{d})$. Using this, it is a simple matter of rewriting to transform (25) into:

$$\text{per}(\mathcal{M}_\lambda) = \prod_{a=1}^{d-1} \Gamma(1 - \frac{a}{d})^{\gamma(\frac{a}{d})},$$

where $\gamma: \mathbb{Z}/d \rightarrow \mathbb{Q}$ is the function

$$\gamma(a) = \frac{2}{d-1} \sum_{j=0}^{2n} (-1)^j \sum_{p+q=j} ph^q(Z, \Omega_Z^p(\log D^{(\lambda)})) + \sum_{\substack{i \in I \\ \langle a_i \rangle = a}} \chi(D_i^\circ).$$

Since, for all $\lambda \in (\mathbb{Z}/d)^\times$, it satisfies

$$\frac{1}{d} \sum_{a \in \mathbb{Z}/d} \gamma(a) \langle a\lambda \rangle = \sum_{j=0}^{2n} (-1)^j \sum_{p+q=j} ph^q(Z, \Omega_Z^p(\log D^{(\lambda)})) + \sum_{i \in I} \frac{\langle a_i \lambda \rangle}{d} \chi(D_i),$$

to conclude the proof it suffices to show that the right hand side agrees with the alternating weighted sum of Hodge numbers predicted by the conjecture.

Proposition 2. *The following relation holds for all $\lambda \in (\mathbb{Z}/d)^\times$:*

$$\sum_{j=0}^{2n} (-1)^j \sum_{p+q=j} ph_\lambda^{p,q}(Y) - \sum_{j=0}^{2n} (-1)^j \sum_{p+q=j} ph^q(Z, \Omega_Z^p(\log D^{(\lambda)})) = \sum_{i \in I} \frac{\langle a_i \lambda \rangle}{d} \chi(D_i^\circ).$$

Proof. Let (\star) denote the right hand side of the equality. By point (b) in lemma 3:

$$(\star) = \sum_{p=0}^n (-1)^p \chi(Z, \mathcal{L}^{(\lambda)^{-1}} \otimes_{\mathcal{O}_Z} \Omega_Z^p(\log D^{(\lambda)})) - \sum_{p=0}^n (-1)^p p \chi(Z, \Omega_Z^p(\log D^{(\lambda)})).$$

The Hirzebruch-Riemann-Roch theorem [Ful97, Cor. 15.2.21] and the multiplicativity of the Chern character yield the equality

$$(26) \quad (\star) = \int_Z [\text{ch}(\mathcal{L}^{(\lambda)^{-1}}) - 1] \cdot \left[\sum_{p=0}^n (-1)^p p \text{ch}(\Lambda^p \Omega_Z^1(\log D^{(\lambda)})) \right] \cdot \text{Td}(TZ).$$

Now, a standard computation of Chern classes (see e.g. the proof of [ST97, Lemma 5.2]) shows that, if \mathcal{F} is a locally free \mathcal{O}_Z -module of rank r , then

$$\sum_{p=0}^r (-1)^p p \text{ch}(\Lambda^r \mathcal{F}) = (-1)^r c_{r-1}(\mathcal{F}) + \text{higher order terms}.$$

Applying this to $\Omega_Z^1(\log D^{(\lambda)})$, one sees that the only non-zero contribution to (26) is

$$\int_Z c_1(\mathcal{L}^{(\lambda)^{-1}}) \cdot (-1)^n c_{n-1}(\Omega_Z^1(\log D^{(\lambda)})).$$

On the other hand, $c_1(\mathcal{L}^{(\lambda)^{-1}}) = -\sum \frac{\langle a_i \lambda \rangle}{d} [D_i]$, as follows straightforward from the definition of the sheaves $\mathcal{L}^{(\lambda)}$ ⁷. Therefore,

$$(\star) = \sum_{i \in I} \frac{\langle a_i \lambda \rangle}{d} (-1)^{n-1} \int_Z [D_i] \cdot c_{n-1}(\Omega_Z^1(\log D^{(\lambda)})) = \sum_{i \in I} \frac{\langle a_i \lambda \rangle}{d} \chi(D_i^\circ),$$

where the last equality is the content of Lemma 1. □

This completes the proof of theorem B.

⁷or, alternatively, from (23) and the general expression of Chern classes of vector bundles with logarithmic connections in terms of residues [EV86, App. B].

2.3. Variants. For the application to the proof of Theorem A, we will need a compactly supported variant of the previous results. Let $\pi : Y \rightarrow Z$ be, as before, the cyclic covering ramified at D . Put $U = Z - D$, $E = \pi^{-1}(D)$ and $V = Y - E$. Then there is an isomorphism

$$\mathbb{H}^j(Y, \tilde{\Omega}_Y^\bullet(\log E)) \otimes_{\overline{\mathbb{Q}}} \mathbb{C} \xrightarrow{\sim} H_B^j(V) \otimes_{\mathbb{Q}} \mathbb{C},$$

where $\tilde{\Omega}_Y^\bullet(\log E) = \iota_* \Omega_{Y-\Sigma}^\bullet(\log(E - \Sigma))$ [Ste77, §1.17]. The group G acts on the mixed Hodge structure $H_B^j(V)$ and one can show, as in Lemma 3, that the eigenspace corresponding to λ in the decomposition of $H_B^j(V) \otimes \mathbb{C}$ is isomorphic to

$$\mathbb{H}^j(Z, \mathcal{L}^{(\lambda)^{-1}} \otimes \Omega^\bullet(\log D)) \otimes_{\overline{\mathbb{Q}}} \mathbb{C}.$$

It follows that the alternating product of $P_\lambda(\det_{\mathbb{Q}(\zeta)} H^j(V))$ is given by $\text{per}(\mathcal{M}_\lambda)$, regarded now as an object over U instead of $U^{(\lambda)}$.

Remark 6. *Mutatis mutandi*, the proof of Theorem B shows that

$$\prod_{j=0}^{2n} P_\lambda(\det_{\mathbb{Q}(\zeta)} H^j(V))^{(-1)^j} \sim_{\overline{\mathbb{Q}}^\times} \prod_{a=1}^{d-1} \Gamma(1 - \frac{a}{d})^{\gamma(a/\lambda)}$$

for any function $\gamma : \mathbb{Z}/d \rightarrow \mathbb{Q}$ satisfying

$$(27) \quad \frac{1}{d} \sum_{a \in \mathbb{Z}/d} \gamma(a) \langle a\lambda \rangle = \sum_{j=0}^{2n} (-1)^j \sum_{p+q=j} p h^q(Z, \mathcal{L}^{(\lambda)^{-1}} \otimes_{\mathcal{O}_Z} \Omega_Z^p(\log D)).$$

From this one can deduce an expression for the periods of the cohomology with compact support $H_c^j(V)$. We use the notation $H^{(\lambda)} := D_{\text{red}} - D^{(\lambda)}$.

Corollary 2. *Let $\gamma' : \mathbb{Z}/d \rightarrow \mathbb{Q}$ be any function such that*

$$(28) \quad \frac{1}{d} \sum_{a \in \mathbb{Z}/d} \gamma'(a) \langle a\lambda \rangle = \sum_{p+q=j} (-1)^j p h^q(Z, \mathcal{L}^{(\lambda)^{-1}} \otimes_{\mathcal{O}_Z} \Omega_Z^p(\log D)(-H^{(\lambda)}))$$

for all $\lambda \in (\mathbb{Z}/d)^\times$. Then the following holds for all $\lambda \in (\mathbb{Z}/d)^\times$:

$$\prod_{j=0}^{2n} P_\lambda(\det_{\mathbb{Q}(\zeta)} H_c^j(V))^{(-1)^j} \sim_{\overline{\mathbb{Q}}^\times} \prod_{a=1}^{d-1} \Gamma(1 - \frac{a}{d})^{\gamma'(a/\lambda)}.$$

Proof. By Poincaré duality, $H_c^j(V) \otimes H^{2n-j}(V) \rightarrow H_c^{2n}(V)$ is a perfect pairing. Since G acts trivially on $H_c^{2n}(V)$, it follows that $H_c^j(V)_\lambda$ is Poincaré dual to $H^{2n-j}(V)_{-\lambda}$. Therefore, $P_\lambda(\det_{\mathbb{Q}(\zeta)} H_c^j(V)) \cdot P_{-\lambda}(\det_{\mathbb{Q}(\zeta)} H^{2n-j}(V)) \sim_{\overline{\mathbb{Q}}^\times} (2\pi i)^{n\chi(U)}$, whence:

$$\begin{aligned} \prod_{j=0}^{2n} P_\lambda(\det_{\mathbb{Q}(\zeta)} H_c^j(V))^{(-1)^j} &\sim_{\overline{\mathbb{Q}}^\times} (2\pi i)^{n\chi(U)} \prod_{j=0}^{2n} P_{-\lambda}(\det_{\mathbb{Q}(\zeta)} H^j(V))^{(-1)^{j+1}} \\ &\sim_{\overline{\mathbb{Q}}^\times} (2\pi i)^{n\chi(U)} \prod_{a=1}^{d-1} \Gamma(1 - \frac{a}{d})^{-\gamma(-a/\lambda)}, \end{aligned}$$

for any function γ as in (27). Setting $\gamma'(a) = \frac{2}{d-1} n\chi(U) - \gamma(-a)$, one has

$$\frac{1}{d} \sum_{a \in \mathbb{Z}/d} \gamma'(a) \langle a\lambda \rangle = n\chi(U) - \sum_{p+q=k} (-1)^j p h^q(Z, \mathcal{L}^{(-\lambda)^{-1}} \otimes_{\mathcal{O}_Z} \Omega_Z^p(\log D)).$$

It remains to transform this identity into the one in (28). This follows from

$$h^q(Z, \mathcal{L}^{(-\lambda)^{-1}} \otimes_{\mathcal{O}_Z} \Omega_Z^p(\log D)) = h^{n-q}(Z, \mathcal{L}^{(\lambda)^{-1}} \otimes_{\mathcal{O}_Z} \Omega_Z^{n-p}(\log D)(-H^{(\lambda)})),$$

which is the result of applying Serre duality, taking into account (22) and the isomorphism $\Omega_Z^{n-p}(\log D)(-D_{red}) \simeq \Omega_Z^n$. \square

3. PROOF OF THEOREM A

3.1. The cohomology of equivariant blow-ups. Let X be a smooth, projective variety over a subfield k of \mathbb{C} , together with the action of a finite group G , and let $Y \hookrightarrow X$ be a G -equivariant closed subscheme of codimension r . Denote by $\tau : \tilde{X} := \text{Bl}_Y X \rightarrow X$ the blow-up of X along Y , by $e : E \hookrightarrow \tilde{X}$ the immersion of the exceptional divisor and by $\psi : E \rightarrow Y$ the restriction of τ . By the universal property of blow-ups, the action of G canonically lifts to \tilde{X} and the morphisms τ and e are G -equivariant [IT14, §2.1.11]. Besides, let $\mathcal{O}_E(1)$ be the tautological vector bundle on E .

Lemma 5. *With the above notations, the map*

$$(29) \quad H^j(X) \oplus \bigoplus_{\ell=0}^{r-2} H^{j-2\ell-2}(Y)(-\ell-1) \longrightarrow H^j(\tilde{X})$$

$$(\eta, \kappa_0, \dots, \kappa_{r-2}) \longmapsto \tau^* \eta + e_* \left[\sum_{\ell=0}^{r-2} \psi^*(\kappa_\ell) \cup c_1(\mathcal{O}_E(1))^\ell \right]$$

is a G -equivariant isomorphism, compatible with the comparison isomorphism and, in the Betti case, with the Hodge structures.

Proof. Leaving aside the action of G , this is the realization in de Rham and Betti cohomology of the motivic result [Man68, §9], from which the compatibility with the comparison isomorphism and the Hodge structures follows as well. To prove that (29) is G -equivariant, it suffices to show that $g^* \mathcal{O}_E(1)$ and $\mathcal{O}_E(1)$ are isomorphic for all $g \in G$. Indeed, $\mathcal{O}_E(1)$ is the normal bundle $N_{E/\tilde{X}}$ and the differential of g induces, via the exact sequence $0 \rightarrow TE \rightarrow T\tilde{X}|_E \rightarrow N_{E/\tilde{X}} \rightarrow 0$, such an isomorphism. \square

Corollary 3. *Let X be a smooth, projective variety over $\overline{\mathbb{Q}}$, together with an automorphism g of order $d \geq 2$. Let $Y \hookrightarrow X$ be a smooth g -equivariant closed subscheme such that $g|_Y$ has order strictly less than d , and let \tilde{X} be the blow-up of X along Y . Then the Gross-Deligne conjecture holds for $\det_{\mathbb{Q}(\zeta)} H^j(X)$ if and only if it holds for $\det_{\mathbb{Q}(\zeta)} H^j(\tilde{X})$.*

Proof. The assumption implies that g cannot act on the cohomology of Y with eigenvalue ζ^λ for any $\lambda \in (\mathbb{Z}/d)^\times$. Since (29) is equivariant and respects the comparison isomorphism and the Hodge structures, one has $P_\lambda(\det_{\mathbb{Q}(\zeta)} H_B^j(\tilde{X})) = P_\lambda(\det_{\mathbb{Q}(\zeta)} H_B^j(X))$ and $h_\lambda^{p,q}(\tilde{X}) = h_\lambda^{p,q}(X)$ for all $\lambda \in (\mathbb{Z}/d)^\times$, so both sides of the conjecture coincide. \square

3.1.1. In the rest of the section we assume that G is a cyclic group of order d , generated by an element g . We shall perform a sequence of blow-ups with the goal of making the action of G free outside a G -strict normal crossing divisor D . Recall that this means that, for each irreducible component D_i , either $gD_i = D_i$ or $gD_i \cap D_i = \emptyset$. Given a subgroup H of G , let X^H denote the (closed) subscheme of fixed points of H . We will make repeated use of the fact that X^H is smooth [Edi92, Prop. 3.1].

Let $d = p_1^{e_1} \cdots p_r^{e_r}$ be the prime factorization of d . Put $N = \sum_{i=1}^r e_i$ and let S denote the set of divisors of d . We can write S as a disjoint union of the subsets

$$S_\ell = \{p_1^{f_1} \cdots p_r^{f_r} \mid 0 \leq f_i \leq e_i, \sum_{i=1}^r f_i = \ell\}$$

for $\ell = 0, \dots, N$. For example, $S_0 = \{1\}$ and $S_1 = \{p_1, \dots, p_r\}$. Given an integer s in S , let H_s be the subgroup of G generated by g^s . We construct a tower of N blow-ups

$$\tilde{X} := X_{N-1} \xrightarrow{\tau_{N-1}} \dots \xrightarrow{\tau_1} X_0 \xrightarrow{\tau_0} X_{-1} := X$$

as follows: $\tau_0 : X_0 \rightarrow X$ is the blow-up of X along X^G and, for $m \geq 1$, $\tau_m : X_m \rightarrow X_{m-1}$ is the blow-up of X_{m-1} along the strict transform of $\bigcup_{s \in S_m} X^{H_s}$ by $\tau_{m-1} \circ \dots \circ \tau_0$. Denote by $\tau : \tilde{X} \rightarrow X$ the composition $\tau_{N-1} \circ \dots \circ \tau_0$ and by E the inverse image of $\bigcup_{s \in S} X^{H_s}$ by τ .

Lemma 6.

- (a) E is a G -strict normal crossings divisor and G freely on $\tilde{X} - E$.
- (b) G acts with order strictly less than d on the cohomology of E .
- (c) The Gross-Deligne conjecture holds for $\det_{\mathbb{Q}(\zeta)} H^j(X)$ if and only if it holds for $\det_{\mathbb{Q}(\zeta)} H^j(\tilde{X})$.

Proof. Assertion (a) is proved in [IT14, Lemma 4.2.3]. Point (b) follows from the fact that \tilde{X} is obtained by successively blowing-up G -invariant smooth closed subschemes where G acts with order strictly less than d , and (c) follows from this and Corollary 3. \square

3.2. End of the proof. Fix an integer $d \geq 2$ and a primitive d -th root of unity ζ . Let X be a smooth, projective variety over $\overline{\mathbb{Q}}$ acted upon by an automorphism g of order d . By lemma 6, for the purpose of proving theorem A, we can assume that X contains a G -strict normal crossings divisor E such that g acts freely on $V = X - E$ and that the action of g on the cohomology of E has order strictly less than d . Then one has a G -equivariant commutative diagram with exact horizontal arrows and vertical isomorphisms

$$(30) \quad \begin{array}{ccccccc} H_{dR}^{j-1}(E) \otimes_{\overline{\mathbb{Q}}} \mathbb{C} & \longrightarrow & H_{dR,c}^j(V) \otimes_{\overline{\mathbb{Q}}} \mathbb{C} & \longrightarrow & H_{dR}^j(X) \otimes_{\overline{\mathbb{Q}}} \mathbb{C} & \longrightarrow & H_{dR}^j(E) \otimes_{\overline{\mathbb{Q}}} \mathbb{C} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H_B^{j-1}(E) \otimes_{\mathbb{Q}} \mathbb{C} & \longrightarrow & H_{B,c}^j(V) \otimes_{\mathbb{Q}} \mathbb{C} & \longrightarrow & H_B^j(X) \otimes_{\mathbb{Q}} \mathbb{C} & \longrightarrow & H_B^j(E) \otimes_{\mathbb{Q}} \mathbb{C}. \end{array}$$

Since G acts on the cohomology of E with order strictly less than d , one has

$$P_\lambda(\det_{\mathbb{Q}(\zeta)} H_B^j(X)) = P_\lambda(\det_{\mathbb{Q}(\zeta)} H_{B,c}^j(V))$$

for all $\lambda \in (\mathbb{Z}/d)^\times$. To compute the right hand side, we observe that the quotient of V by the action of G is now a smooth quasi-projective variety U , and that the quotient map $V \rightarrow U$ is an unramified cyclic cover of U . Let Z be a smooth projective variety over $\overline{\mathbb{Q}}$, containing U as the complement of a normal crossings divisor D_{red} , with irreducible components D_i . Let $\pi : Y \rightarrow Z$ be the normalization of X in the fraction field of U . Then $\pi : Y \rightarrow Z$ is isomorphic to a cyclic cover constructed out of a line bundle \mathcal{L} and a divisor D with support D_{red} by the procedure described in §2.1. Thus, the statement follows from Corollary 2 and from the fact that the horizontal arrows in the second line of (30) are morphisms of Hodge structures.

APPENDIX A. PERIODS OF THE UNIT OBJECT

The aim of this appendix is to prove Proposition 1, saying that the determinant of periods of the unit object is an explicit power of $2\pi i$, up to some factor of degree at most two over the base field. As before, U will be the complement of a normal crossings

divisor D on a smooth, projective variety X , of dimension n , over a subfield k of \mathbb{C} . The comparison isomorphism defines rank one objects

$$[\det H^j(U)] = (\det H_{dR}^j(U/k), \det H^j(U_{\mathbb{C}}^{\text{an}}, \mathbb{Q}), \det \rho^j)$$

in $MPic_{k, \mathbb{Q}}(\text{Spec}(k)) \simeq \mathbb{C}^\times / k^\times$, and $\text{per}(\mathbf{1}_U)$ is nothing other than $\bigotimes_{j=0}^{2n} [\det H^j(U)]^{(-1)^j}$. Using mixed Hodge theory, we will reduce the computation to the projective case.

A.1. The compact case. Assume first that D is empty, so $U = X$. It clearly suffices to treat the case where X is connected. Choose a hyperplane section H of X and consider the classes $[H]_{dR} \in H_{dR}^2(X/k)$ and $[H]_B \in H^2(U_{\mathbb{C}}^{\text{an}}, \mathbb{Q})$ in the Rham and singular cohomology respectively. By construction of the first Chern class, ρ^2 sends $[H]_{dR} \otimes_k 1_{\mathbb{C}}$ to $(2\pi i)[H]_B \otimes_{\mathbb{Q}} 1_{\mathbb{C}}$ (see e.g. [Bos13, Lemma 2.1]). Since $H^{2n}(X)$ is generated by $[H]^n$ and the comparison isomorphism commutes with the algebra structure on cohomology, this implies that $[\det H^{2n}(X)] = (2\pi i)^n$ in $\mathbb{C}^\times / k^\times$. Besides, by Poincaré duality, the pairing $H^j(X) \otimes H^{2n-j}(X) \rightarrow H^{2n}(X)$ is non-degenerate both on de Rham and Betti cohomology, and compatible with the comparison isomorphism. It follows that

$$(31) \quad [\det H^j(X)] \otimes [H^{2n-j}(X)] = (2\pi i)^{nb_j(X)},$$

where $b_j(X)$ stands for the j -th Betti number of X . From here one deduces at once:

Lemma 7. *The relation $\text{per}(\mathbf{1}_X)^2 = (2\pi i)^{n\chi(X)}$ holds in $\mathbb{C}^\times / k^\times$.*

One can actually prove the following stronger result:

Lemma 8. *The relation $[\det H^j(X)]^2 = (2\pi i)^{jb_j(X)}$ holds in $\mathbb{C}^\times / k^\times$.*

Proof. In view of the identities $[\det H^{2n}(X)] = (2\pi i)^n$ and (31), it is enough to prove the statement for $0 < j \leq n$. Consider the bilinear forms $\langle \cdot, \cdot \rangle_{dR} : H_{dR}^j(X/k) \otimes H_{dR}^j(X/k) \rightarrow k$ and $\langle \cdot, \cdot \rangle_B : H^j(X_{\mathbb{C}}^{\text{an}}, \mathbb{Q}) \otimes H^j(X_{\mathbb{C}}^{\text{an}}, \mathbb{Q}) \rightarrow \mathbb{Q}$ defined by:

$$\alpha \cup \beta \cup [H]^{n-j} = \langle \alpha, \beta \rangle \cdot [H]^n.$$

By Poincaré duality and the hard Lefschetz theorem, they are both non-degenerate. After extension of scalars, ρ^j sends $\langle \cdot, \cdot \rangle_{dR}$ to $(2\pi i)^{-j} \langle \cdot, \cdot \rangle_B$, so letting A_{dR}, A_B and P be matrices of the bilinear forms and the comparison isomorphism with respect to rational basis, one has $A_{dR} = (2\pi i)^j P^t A_B P$, and the result follows by taking determinants. \square

A.2. The open case. Assume now that the divisor D is not empty and denote by $\{D_i\}_{i \in I}$ its irreducible components. For each $0 \leq m \leq |I|$, let $D(m)$ be the disjoint union of all schematic intersections $D_J := \bigcap_{i \in J} D_i$ indexed by a subset $J \subset I$ of cardinal m ; it is a smooth scheme of dimension $n - m$. Let $a_m : D(m) \rightarrow X$ be the morphism deduced from the inclusions $D_J \hookrightarrow X$. Notice that, although the D_i need not be k -rational, the $D(m)$ are. Recall from [Del71, §3.1] that the “order of poles” provides the complex of logarithmic differentials $\Omega_X^\bullet(\log D)$ with an increasing filtration W and that the Poincaré residue induces an isomorphism of complexes

$$(32) \quad \text{Gr}_m^W \Omega_X^\bullet(\log D) \xrightarrow{\sim} a_{m*} \Omega_{D(m)}^\bullet[-m].$$

Through the isomorphism $\mathbb{H}^j(X, \Omega_X^\bullet(\log D)) \otimes_k \mathbb{C} \xrightarrow{\sim} H^j(U_{\mathbb{C}}^{\text{an}}, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}$, W induces the so-called weight filtration, which is defined over \mathbb{Q} . Thanks to (32), one gets the following decomposition in K_0 of the category $M_{k, \mathbb{Q}}(\text{Spec}(k))$ (see [PS08, p. 99])

$$(33) \quad \sum_{j=0}^{2n} (-1)^j [H^j(U)] = \sum_{m \geq 0} (-1)^m \sum_{\ell=0}^{2(n-m)} (-1)^\ell [H^\ell(D(m))(-m)].$$

A.2.1. *Proof of Proposition 1.* From (33) one sees that

$$\text{per}(\mathbf{1}_U) = \prod_{m \geq 0} [\text{per}(\mathbf{1}_{D(m)}) \cdot (2\pi i)^{m\chi(D(m))}]^{(-1)^m}.$$

In particular, taking into account Lemma 7, one gets

$$\text{per}(\mathbf{1}_U)^2 = (2\pi i)^{\sum (-1)^m (n+m)\chi(D(m))},$$

so to conclude it suffices to prove the following completely standard

Lemma 9.

$$\frac{1}{2} \sum_{m \geq 0} (-1)^m (n+m) \chi(D(m)) = \sum_{k=0}^{2n} (-1)^k \sum_{p+q=k} ph^q(X, \Omega_X^p(\log D)).$$

Proof. Using the isomorphism (32), one has

$$\begin{aligned} \sum_{k=0}^{2n} (-1)^k \sum_{p+q=k} ph^q(X, \Omega_X^p(\log D)) &= \sum_{m \geq 0} \sum_{k=0}^{2n} (-1)^k \sum_{p+q=k} ph^q(X, \text{Gr}_m^W \Omega_X^p(\log D)) \\ &= \sum_{m \geq 0} \sum_{k=0}^{2n} (-1)^k \sum_{p+q=k} ph^q(D(m), \Omega_{D(m)}^{p-m}(m)) \\ &= \sum_{m \geq 0} (-1)^m \sum_{k=0}^{2(n-m)} (-1)^k \sum_{p+q=k} (p+m) h^q(D(m), \Omega_{D(m)}^p). \end{aligned}$$

On the other hand, $\chi(D(m)) = \sum (-1)^k \sum h^q(D(m), \Omega_{D(m)}^p)$ by degeneration of the Hodge-de Rham spectral sequence, so the assertion follows if we show that

$$\frac{n-m}{2} \chi(D(m)) = \sum_{k=0}^{2(n-m)} (-1)^k \sum_{p+q=k} ph^q(D(m), \Omega_{D(m)}^p)$$

for all m . But this is only a matter of rewriting once we know, by Serre duality, that the numbers $h^q(D(m), \Omega_{D(m)}^p)$ and $h^{n-m-q}(D(m), \Omega_{D(m)}^{n-m-p})$ are equal. \square

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